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Equivalence of local potential approximations

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Abstract: In recent papers it has been noted that the local potential approximation of the Legendre and Wilson-Polchinski flow equations give, within numerical error, identical results for a range of exponents and Wilson-Fisher fixed points in three dimensions, providing a certain "optimised" cutoff is used for the Legendre flow equation. Here we point out that this is a consequence of an exact map between the two equations, which is nothing other than the exact reduction of the functional map that exists between the two exact renormalization groups. We note also that the optimised cutoff does not allow a derivative expansion beyond second order.

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The fundamentals and applications of the "exact renormalization group" (exact RG), discovered and so christened independently by Wilson and Wegner [1, 2], have been studied intensively since the beginning of the nineties [3 − 5]. The central reason for this recrudescence is the general acceptance that, far from being merely formal exact realizations of Wilson's RG ideas, these ideas form the basis for powerful and flexible approximations in non-perturbative quantum field theory. (For reviews, see for example refs. [6].)

The two most widely used realizations of such exact RGs (for others see [7, 8]), are Polchinski's version [9], equivalent to Wilson's [1] by a change of variables [10, 11], and the version for the Legendre effective action [3 − 5]. We will be interested in the case where these are applied to $\mathrm{O}(N)$ invariant N-component real scalar field theory in D Euclidean dimensions.

For such a theory, Polchinski's version is given by:[1]

$$\frac{\partial S_\Lambda}{\partial \Lambda} = \frac{1}{2}\frac{\delta S_\Lambda}{\delta \Phi_a} \cdot \frac{\partial \Delta_{\mathrm{UV}}}{\partial \Lambda} \cdot \frac{\delta S_\Lambda}{\delta \Phi_a} - \frac{1}{2}\,\mathrm{tr}\,\frac{\partial \Delta_{\mathrm{UV}}}{\partial \Lambda} \cdot \frac{\delta^2 S_\Lambda}{\delta \Phi \delta \Phi}\,, \tag{1}$$

where $\Phi_a(x)$ is the N-component scalar field and $S_\Lambda[\Phi]$ is the interaction part of the wilsonian effective action

$$S^{\mathrm{eff}}_\Lambda = \frac{1}{2}\Phi_a.\Delta^{-1}_{\mathrm{UV}}.\Phi_a + S_\Lambda\,. \tag{2}$$

Λ is the effective cutoff, $\Delta_{\mathrm{UV}}(q,\Lambda) = C_{\mathrm{UV}}(q^2/\Lambda^2)/q^2$ is the ultraviolet regularised propagator, and C_{UV} the ultra-violet cutoff function.

On the other hand, the flow equation for the Legendre effective action, also called effective average action, is given by:

$$\frac{\partial}{\partial \Lambda}\Gamma_\Lambda[\varphi] = -\frac{1}{2}\;\mathrm{tr}\;\frac{1}{\Delta_{\mathrm{IR}}}\frac{\partial \Delta_{\mathrm{IR}}}{\partial \Lambda} \cdot A^{-1}\,, \tag{3}$$

where

$$A_{ab} = \delta_{ab} + \Delta_{\mathrm{IR}} \cdot \frac{\delta^2 \Gamma_\Lambda}{\delta \varphi_a \delta \varphi_b}. \tag{4}$$

Here $\Gamma_\Lambda[\varphi]$ is the interaction part of the Legendre effective action

$$\Gamma^{tot}_\Lambda = \frac{1}{2}\varphi_a \cdot \Delta^{-1}_{\mathrm{IR}} \cdot \varphi_a + \Gamma_\Lambda[\varphi], \tag{5}$$

where the propagator has been replaced by an infrared regularised propagator $\Delta_{\mathrm{IR}}(q,\Lambda) = C_{\mathrm{IR}}(q^2/\Lambda^2)/q^2$, C_{IR} being the infrared cutoff function.

One of the simplest and most powerful approximations, which is also widely used, is the Local Potential Approximation (LPA) [13]. In this case one simply makes the model approximation that the above actions are of the form of a potential only, and discards all parts of the right hand sides of (1) and (3) that do not fit this approximation. More rigorously, providing the cutoff functions are smooth, the actions have a derivative expansion to all orders [10], and the LPA simply amounts to taking the lowest order in this expansion, setting all higher order terms to zero.

[1] We base our notation on refs. [12, 5] for reasons that will become clear.

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For the LPA of the Polchinski equation (1), it turns out that by changes of variables all explicit cutoff function dependence disappears and thus the LPA yields universal results [14]. For the Legendre flow equation we adopt the "optimised" infrared cutoff that Litim has advocated. It is defined through an additive infrared cutoff which replaces the propagator $1/q^2$ by $1/(q^2 + C_{add})$, and is defined by [15]

$$C_{\rm add} = (\Lambda^2 - q^2)\theta(\Lambda^2 - q^2)\,. \tag{6}$$

Thus

$$\begin{aligned} \Delta_{\rm IR} &= \frac{1}{\Lambda^2} \qquad \text{for} \qquad q < \Lambda\,, \\ \Delta_{\rm IR} &= frac1q^2 \qquad \text{for} \qquad q > \Lambda\,. \end{aligned} \tag{7}$$

Writing $t = \ln(\mu/\Lambda)$, μ some arbitrary physical mass scale, $S_\Lambda = \int d^D x\, U(y,\Lambda)$ and $\Gamma_\Lambda = \int d^D x\, V(z,\Lambda)$, where $y = \Phi^a\Phi_a$ and $z = \varphi^a\varphi_a$, and scaling to dimensionless variables using Λ whilst also absorbing some constants, the LPA approximations can then be written as [14–16]:[2]

$$\partial_t U(y,t) + (D-2)yU' - DU = -4y(U')^2 + 2NU' + 4yU'' \tag{8}$$

$$\partial_t V(z,t) + (D-2)zV' - DV = -\frac{N-1}{1+2V'} - \frac{1}{1+2V'+4zV''}\,. \tag{9}$$

We now come to the central issue of this letter. As noted first in ref. [17], for the Wilson-Fisher fixed point in $D = 3$ dimensions and for various N, several RG eigenvalues[3] computed from eqs. (8), (9) agree to all published digits [17]. This was recently extended by Bervillier [18], and analysed in much greater detail very recently by Litim [16]. The main conclusions are that this is a very surprising result, given the inequivalent derivative expansions and the dependence on cutoff function displayed by the Legendre flow equation even at the LPA level [17]. Most recently Litim conjectures from this "most remarkable [...] high degree of coincidence" that the universal content of (8) and (9) must be the same [16]. Underlining the surprising nature of the coincidence, Litim goes on to argue that, while the derivative expansion is particularly simple to implement in the Wilson-Polchinski equation, the Legendre form yields more stable results even at the LPA level.

And yet, how can two partial differential equations, neither of which is exactly soluble (even at the fixed points) yield such non-trivial agreement, without being fundamentally related? The answer has to be that they are in fact related by a change of variables.

Already long ago, it was shown that as an exact statement, a Legendre transform relation exists between the two functionals Γ_Λ and S_Λ [5], providing only that the cutoff functions satisfy the sum rule

$$C_{\rm IR} + C_{\rm UV} = 1\,. \tag{10}$$

[2]prime is differentiation by y or z as appropriate. These are precisely the equations that appear in ref. [16] up to scaling z and y by $1/2$, a correction of sign and large N scaling in (8).

[3]equivalently exponents, including those for corrections to scaling

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The Legendre transform relation is

$$S_\Lambda[\Phi] = \Gamma_\Lambda[\varphi] + \frac{1}{2}(\varphi_a - \Phi_a) \cdot \Delta_{\mathrm{IR}}^{-1} \cdot (\varphi_a - \Phi_a)\,. \tag{11}$$

It transforms the corresponding flow equations into each other [5]: they are in fact two realizations of the same exact RG.

Furthermore, for *constant* fields Φ and φ, all the higher derivative terms inside the effective actions vanish and this relation collapses to [12]:

$$U(y, \Lambda) = V(z, \Lambda) + \frac{1}{2}\Delta_{\mathrm{IR}}^{-1}(0, \Lambda)(\varphi - \Phi)^2\,. \tag{12}$$

Note well, that (12) is still an exact statement. It is trivially extended to hold for any field theory, not just $\mathrm{O}(N)$ scalar field theory.

On the other hand, once we start to approximate the flow equations (1), (3), relation (12) will in general be broken. Indeed in ref. [12], we showed that for general cutoffs, although the large N limit of the LPA for these two equations yield the known exact results for the RG eigenvalues, only $V(z, t)$ is exact in this limit: the Wilson-Polchinski effective potential $U(y, t)$ is not the correct (i.e. exact) potential for general cutoffs, even in the limit of large N.

It is interesting to note that Litim's optimised cutoff (7) satisfies, in particular in (11),

$$\Delta_{\mathrm{IR}}^{-1} = \Lambda^2 \quad \textit{to all orders in the derivative expansion}\,, \tag{13}$$

(i.e. here to all orders in the Taylor expansion in q^2 about $q = 0$). It is tempting to speculate that this is somehow responsible for the preservation of the Legendre transform relation at the LPA level. Indeed, we will shortly see that (12) does still hold. However, this speculation would then suggest that the relation continues to hold at higher orders of the derivative expansion, which seems very unlikely, and in any case (7) is one of only many cutoff propagators with the property (13), while the functional form of (9) is specific to the choice (7).

On the other hand, (13) and even more importantly the lack of smoothness in (7) (as evidenced by the appearance of Heaviside θ function), mean the derivative expansion must actually break down at some point. We will firm up this observation at the end, where we will see that the derivative expansion breaks down at $\mathrm{O}(\partial^4)$.

One immediate corollary of these observations is that, while (8) is not exact for general cutoffs, even in the large N limit, the large N limit of this equation *is* the exact answer when we use the (rather strange) cutoff following from (7) and (10). This thus provides a simple explanation for why we found that (8) nevertheless gave the correct RG eigenvalues in the large N limit.[4]

Turning now to the proof of equivalence of the two LPAs (8) and (9), we note that the scaled version of (12) is:

$$U(y, t) = V(z, t) + \frac{1}{2}(\varphi - \Phi)^2\,. \tag{14}$$

[4]However, we also showed that a much larger class of incorrect equations nevertheless give the right RG eigenvalues in this limit [12].

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Thus

$$\partial_t|_y U = \partial_t|_z V \,, \tag{15}$$

a standard consequence of Legendre transforms, and equally

$$\varphi_a - \Phi_a = 2\Phi_a U' = -2\varphi_a V' \,. \tag{16}$$

From this it follows that φ_a and Φ_a point in the same direction and thus [12]

$$\sqrt{\frac{z}{y}} = 1 - 2U' = \frac{1}{1+2V'} \,. \tag{17}$$

This relation deals with the first term on the right hand side of (9). On the other hand, it also implies $U'\sqrt{y} = V'\sqrt{z}$ and thus $zV' = yU'(1-2U')$. Squaring (16), we have $U - V = 2y(U')^2$, so

$$(D-2)zV' - DV = (D-2)yU' - DU + 4y(U')^2 \,. \tag{18}$$

Note that this concurs with the demonstration of equivalence of (1) and (3) [5]: there, differentiating (11) generates a term that cancels the tree level contribution, i.e. the first term on the right hand side of (1). After scaling, it is the D dependent terms (generated by the trivial Λ dependence) that play the same rôle.

Differentiating (17) by y and comparing to differentiating the inverse by z, one finds

$$1 - 2U' - 4yU'' = \frac{1}{1+2V'+4zV''} \,. \tag{19}$$

Finally, using this, (15), (17) and (18), and making the standard discard of the constant vacuum energy contribution (here $-N$) [9, 5], it is easy to see that (8) and (9) are indeed Legendre transforms of each other under the map (14), as claimed.

Although we established the equivalence in $\mathrm{O}(N)$ invariant scalar field theory, so as to make immediate contact with refs. [16 − 18], it is clear that the $\mathrm{O}(N)$ invariance is not crucial and this equivalence would follow similarly, at least for any scalar field theory. Of course it follows that universal information, such as the RG eigenvalues around any fixed point, must agree between the two realizations, here (8) and (9). But more than that, the relations, e.g. (17), provide an explicit map between the solutions of each equation. It is therefore only necessary to solve one of them, after which the change of variables can be performed to get the other. Both the solution and the change of variables can be found numerically. An obvious but important consequence is that for the optimised cutoff (7), the LPA is just as accurate for all quantities irrespective of whether we use the Wilson-Polchinski or the Legendre form as the starting point.

We finish by completing our remark on the limitations of (7) when considering derivative expansions. A sharp regulator is known to break down already at $\mathrm{O}(\partial^2)$ [5, 19]. In its additive form (6), Litim's optimised regulator is in fact the first integral of a sharp cutoff. We should then expect that this regulator survives at $\mathrm{O}(\partial^2)$, but breaks down at $\mathrm{O}(\partial^4)$. We now confirm this.

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We can investigate the properties of momentum expansions to high order if we first focus on perturbative contributions [20]. Indeed, as in that paper let us focus on the Legendre effective action one-loop four-point vertex in $D = 4$ dimensional $\lambda\varphi^4$ theory. The flow of this vertex is the sum of three contributions (the s, t and u channels) of the form:

$$-\lambda^2 \int \frac{d^4q}{(2\pi)^4}\, \Delta_{\mathrm{IR}}(q+p,\Lambda)\, \Lambda \frac{\partial}{\partial\Lambda} \Delta_{\mathrm{IR}}(q,\Lambda)\,. \tag{20}$$

Using (7) and writing $(p+q)^2 = p^2 + q^2 + 2pqx$, where x is the cosine of the angle between p^μ and q^μ, and scaling out Λ, the integral is proportional to

$$\int_0^1 dq \int_{-1}^1 dx\ q^3 \left\{ \left(1 - \frac{1}{(p+q)^2}\right) \theta\left[1 - (p+q)^2\right] + \frac{1}{(p+q)^2} \right\}. \tag{21}$$

We are not interested in the proportionality constant, namely $2\lambda^2/(4\pi)^2$. Since we will be expanding in the external momentum p, we can assume it is small. The integral over x is then straightforward; the θ function is relevant only when $q > 1-p$, splitting the q integral into two domains. Thus we obtain the contribution as

$$\frac{1}{3}\left(p^2 + \frac{1}{p}\right) \ln(1+p) + \frac{1}{6}(1+p) - \frac{4}{9}p^2 - \frac{1}{60}p^4 \tag{22}$$

which expands as

$$\frac{1}{2} - \frac{1}{3}p^2 + \frac{1}{4}p^3 - \frac{7}{60}p^4 + \frac{1}{18}p^5 + \cdots\,. \tag{23}$$

If this corresponded to a derivative expansion, the powers in p would all be even, however we see that as expected odd powers appear after $\mathrm{O}(p^2)$. Thus beyond $\mathrm{O}(\partial^2)$ the cutoff (6), equivalently (7), must be treated in terms of the more general momentum scale expansion developed in refs. [5, 19]. However, we would also expect that, in common with sharp cutoffs, the momentum scale expansion does not have good convergence properties [20].

References

[1] K. Wilson and J. Kogut, *The renormalization group and the epsilon expansion*, *Phys. Rept.* **12** (1974) 75.

[2] F.J. Wegner and A. Houghton, *Renormalization group equation for critical phenomena*, *Phys. Rev.* **A 8** (1973) 401.

[3] C. Wetterich, *Exact evolution equation for the effective potential*, *Phys. Lett.* **B 301** (1993) 90.

[4] M. Bonini, M. D'Attanasio and G. Marchesini, *Perturbative renormalization and infrared finiteness in the Wilson renormalization group: the massless scalar case*, *Nucl. Phys.* **B 409** (1993) 441 [hep-th/9301114].

[5] T.R. Morris, *The exact renormalization group and approximate solutions*, *Int. J. Mod. Phys.* **A 9** (1994) 2411 [hep-ph/9308265].

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[6] T.R. Morris, *Elements of the continuous renormalization group*, *Prog. Theor. Phys. Suppl.* **131** (1998) 395–414 [hep-th/9802039];
C. Bagnuls and C. Bervillier, *Exact renormalization group equations: an introductory review*, *Phys. Rept.* **348** (2001) 91 [hep-th/0002034];
J. Berges, N. Tetradis and C. Wetterich, *Non-perturbative renormalization flow in quantum field theory and statistical physics*, *Phys. Rept.* **363** (2002) 223 [hep-ph/0005122].

[7] J.I. Latorre and T.R. Morris, *Exact scheme independence*, *JHEP* **11** (2000) 004 [hep-th/0008123]; *Scheme independence as an inherent redundancy in quantum field theory*, *Int. J. Mod. Phys.* **A 16** (2001) 2071 [hep-th/0102037].

[8] S. Arnone, A. Gatti and T.R. Morris, *A proposal for a manifestly gauge invariant and universal calculus in Yang-Mills theory*, *Phys. Rev.* **D 67** (2003) 085003 [hep-th/0209162].

[9] J. Polchinski, *Renormalization and effective lagrangians*, *Nucl. Phys.* **B 231** (1984) 269.

[10] T.R. Morris, *Derivative expansion of the exact renormalization group*, *Phys. Lett.* **B 329** (1994) 241 [hep-ph/9403340].

[11] G.R. Golner, *Exact renormalization group flow equations for free energies and N-point functions in uniform external fields*, hep-th/9801124;
C. Bervillier, *The Wilson-Polchinski exact renormalization group equation*, *Phys. Lett.* **A 332** (2004) 93 [hep-th/0405025].

[12] M. D'Attanasio and T.R. Morris, *Large-N and the renormalization group*, *Phys. Lett.* **B 409** (1997) 363 [hep-th/9704094].

[13] J.F. Nicoll, T.S. Chang and H.E. Stanley, *Approximate renormalization group based on the Wegner-Houghton differential generator*, *Phys. Rev. Lett.* **33** (1974) 540.

[14] R.D. Ball, P.E. Haagensen, I. Latorre, Jose and E. Moreno, *Scheme independence and the exact renormalization group*, *Phys. Lett.* **B 347** (1995) 80 [hep-th/9411122];
J. Comellas and A. Travesset, O(N) *models within the local potential approximation*, *Nucl. Phys.* **B 498** (1997) 539 [hep-th/9701028].

[15] D.F. Litim, *Optimised renormalisation group flows*, *Phys. Rev.* **D 64** (2001) 105007 [hep-th/0103195].

[16] D.F. Litim, *Universality and the renormalisation group*, hep-th/0503096.

[17] D.F. Litim, *Mind the gap*, *Int. J. Mod. Phys.* **A 16** (2001) 2081 [hep-th/0104221].

[18] C. Bervillier, O(N) *Wilson-Polchinski exact renormalization group equation: leading and next-to-leading orders in the derivative expansion*, hep-th/0501087.

[19] T.R. Morris, *Momentum scale expansion of sharp cutoff flow equations*, *Nucl. Phys.* **B 458** (1996) 477 [hep-th/9508017].

[20] T.R. Morris and J.F. Tighe, *Convergence of derivative expansions of the renormalization group*, *JHEP* **08** (1999) 007 [hep-th/9906166]; *Convergence of derivative expansions in scalar field theory*, *Int. J. Mod. Phys.* **A 16** (2001) 2095 [hep-th/0102027].